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"Transportation problems and the Arens-Eells embedding for non-archimedean fields"

This project is joint with Menachem Shlossberg (Reichman University, Israel) and is based on our joint work [2]. We study a non-archimedean (NA) version of transportation problems and introduce some related constructions. Among others :

- *NA version of the Arens-Eells construction* ;
- naturally arising ultra-norms which we call *Kantorovich ultra-norms* ;
- *NA G-value (generalized integer value) property* ;
- *free NA locally convex spaces.*

Theorem. *For every NA valued field (e.g., the field $\mathbb{Q}_p$ of p-adic numbers) and every ultra-metric space the naturally defined inf-max cost formula for NA transportation problem achieves its infimum. This means that the NA transportation problem always has an optimal solution.*

We present an NA version of the Arens-Eells embedding and introduce the naturally arising Kantorovich ultra-(semi)norms, defined for an ultra-(pseudo)metric space (X, d) on free vector spaces $L_{\mathbb{F}}(X)$ via min-max formula. For an arbitrary NA valued field $\mathbb{F}$ and for any $u \in L_{\mathbb{F}}(X)$ the value of the Kantorovich ultra-(semi)norm $||u||$ can be approximated as

$$||u|| = \inf \left\{ \max_{1\leq i\leq k} |s_i| d(x_i, y_i) : u = \sum_{i=1}^{k} s_i(x_i - y_i),\ x_i, y_i \in \operatorname{supp}(u),\ s_i \in \mathbb{F} \right\}.$$

Note that the analogous property in the archimedean case does not hold in general (i.e., for $\mathbb{F} = \mathbb{C}$, the field of complex numbers), in contrast to the case $\mathbb{F} = \mathbb{R}$.

The infimum here is, in fact, a **minimum**. This refinement, which comes from Min-attaining Theorem provides another contrast to the archimedean case. Indeed, the infimum is not attained for $\mathbb{F} = \mathbb{Q}(i)$. Another refinement concerns the coefficients (the *G-value property*), that is, it is enough to take the coefficients from the additive subgroup G_u of $\mathbb{F}$ generated by the *normal coefficients* λ_i of $u = \sum_{i=1}^{n} \lambda_i x_i$. Namely, we show that

$$||u|| = \min \left\{ \max_{1\leq i,j\leq m} |c_{ij}| d(x_i, x_j) : c_{ij} \in \mathbb{F},\ \forall i : 1 \leq i \leq n\ \sum_{j=1}^{m} c_{ij} - \sum_{j=1}^{m} c_{ji} = \lambda_i \right\},$$

such that all coefficients c_{ij} belong to G_u. As a particular case we get an NA generalization of the so-called *integer value property* (well known in case $\mathbb{F} = \mathbb{R}$).

We introduce the *free NA locally convex spaces* for NA uniform spaces and describe their topologies in terms of Kantorovich ultra-seminorms. Using some ideas from [1] we show that for an ultra-metric space (X, d) and a trivially valued field $\mathbb{F}$, the free NA locally convex space $L_{\mathbb{F}}(X, \mathcal{U}(d))$ (of the uniformity $\mathcal{U}(d)$ of d) is even *normable* by the Kantorovich ultra-norm.

For a nontrivially valued NA field $\mathbb{F}$, the dual NA Banach space of $L^0_{\mathbb{F}}(X)$ can be identified with the NA Banach space Lip_0 of all pointed Lipschitz functions $f : (X, x_0) \to \mathbb{F}$.

Using Ostrowski's classical theorem we prove that in case $\mathbb{F}$ is an NA valued field of zero characteristic, the uniform free NA abelian topological group $A_{\mathcal{NA}}(X, \mathcal{U})$ is a topological subgroup of $L_{\mathbb{F}}(X, \mathcal{U})$ if and only if the restricted valuation on $\mathbb{Q}$ is trivial. For example, this is the case for the Levi-Civita field.

We are going to discuss some possible developments and problems.

Références

[1] M. Megrelishvili and M. Shlossberg, *Free non-archimedean topological groups*, Comment. Math. Univ. Carolin. **54 :2** (2013), 273-312.

[2] M. Megrelishvili and M. Shlossberg, *Non-Archimedean Transportation Problems and Kantorovich Ultra-Norms*, P-Adic Numbers, Ultrametric Analysis, and Applications, 8 (2016), 125-148.