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"Lipschitz-free spaces, Lipschitz spaces and de Leeuw representations"

Let (M, d) be a complete metric space with base point 0, define the completely metrisable space $\widetilde{M} = \{(x, y) \in M \times M \,:\, x \neq y\}$, and denote by $\beta\widetilde{M}$ its Stone-Čech compactification. The *de Leeuw transform* $\Phi : \mathrm{Lip}_0(M) \to C(\beta\widetilde{M})$ is given by

$$\Phi f(x, y) = \frac{f(x) - f(y)}{d(x, y)}, \quad f \in \mathrm{Lip}_0(M),\ (x, y) \in \widetilde{M},$$

and extending continuously to $\beta\widetilde{M}$. This is a linear isometric embedding whose adjoint $\Phi^* : C(\beta\widetilde{M})^* \to \mathrm{Lip}_0(M)^*$ is a quotient map.

My coauthors and I have embarked on a systematic and ongoing study of the de Leeuw transform. Given $\psi \in \mathrm{Lip}_0(M)^*$, there exist positive Radon measures μ on $\beta\widetilde{M}$ such that $\Phi^*\mu = \psi$ and $\|\mu\| = \|\psi\|$. Such measures μ we call *optimal (de Leeuw) representations* of ϕ. If such a μ is concentrated on $\widetilde{M}$, then $\Phi^*\mu$ is said to be a *convex integral of molecules*; in this case $\Phi^*\mu \in \mathcal{F}(M)$ and behaves nicely. However, not every element of $\mathcal{F}(M)$ (or any element of $\mathrm{Lip}_0(M)^* \setminus \mathcal{F}(M)$) is a convex integral of molecules. We extend the theory of optimal representations to include such elements, with a view to gleaning further information about the structure of $\mathcal{F}(M)$ and $\mathrm{Lip}_0(M)^*$.

To this end, we present the *Lipschitz realcompactification* $(M^{\mathcal{R}}, \tau)$ of M, a natural τ-lower semicontinuous metric extension $\bar{d}$ of d to M, and explore the fundamental properties of $(M^{\mathcal{R}}, \bar{d}, \tau)$. We compare $\bar{d}$ with the continuous extension of d as a function on $\widetilde{M}$ to $\beta\widetilde{M}$ and classify elements $\zeta \in \beta\widetilde{M}$ according to the value of $\|\Phi^*\delta_\zeta\|$. We conclude by establishing a relationship between the sets on which optimal representations can be concentrated and the key concept of cyclical monotonicity from optimal transport theory, particularly in the context of functionals in $\mathrm{Lip}_0(M)^*$ that 'avoid infinity'.

This is joint work with Ramón Aliaga (Universitat Politècnica de València) and Eva Pernecká (Czech Technical University, Prague).